Technical Memorandum TM-67-15

SOME PROPERTIES OF RAYLEIGH DISTRIBUTED RANDOM VARIABLES AND OF THEIR SUMS AND PRODUCTS

By

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7 April 1967

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SUMMARY

Physical phenomena which can be described by the Rayleigh density function exist in many areas of study; e.g., noise theory, lethality, radar return, etc. This report gives some of the basic properties of the Rayleigh prcbability density function. It includes the functional relationships between the various parameters and graphical displays of these relationships. The density functions for the sum and the product of two Rayleigh distributed random variables and the relationships between their various parameters are described. Illustrations of the use of the density functions are given. Also included are complete tables of the Rayleigh density function and distribution function.

INTRODUCTION

In nature, physical phenomena in many areas of fields of science (for example, noise theory, lethality, radar return, etc.) have amplitude distributions which can be characterized by the Rayleigh density function or some function which can be derived from the Rayleigh density function. Because a literature search failed to turn up any major source of material on the Rayleigh density function, this report was written to fill the need of those persons working with this density function to serve as a quick reference which would describe this density function and some of its properties. The density functions for the sum and product of two Rayleigh distributed random variables and the relationships between their various parameters are described.

The main text is devoted to the properties of the Rayleigh density function and examples of applications. Derivations of the functional relationships are given in the appendix.

ONE RAYLEIGH DISTRIBUTED RANDOM VARIABLE

Definition of Rayleigh Density Function

A random variable X is said to have a Rayleigh density function $p_R(x)$ if the probability density function is of the form

$$p_R(x) = \begin{cases} 0 & \text{if } x < 0 \\ \dfrac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}} & \text{if } x \geq 0 \end{cases} \tag{1}$$

where α is a convenient parameter. $p_R(x)$ is illustrated in figure 1 and tabulated in table 1.

*Rayleigh Distribution Function**

The Rayleigh distribution function $P_R(x)$ is given by

$$P_R(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\frac{x^2}{2\alpha^2}} & \text{if } x \geq 0 \end{cases} \tag{2}$$

This function is illustrated in figure 2 in units of α and tabulated in table 2.

Elucidation of the Rayleigh Parameter (α)

As mentioned above, α is merely a convenient parameter and not the standard deviation for a Rayleigh distributed random variable. It is simply the normalizing factor which is universally used for this function. Its utility will be made evident in the following paragraphs.

* Current terminology will be followed with respect to the names density function, and distribution function. Some authors call the former the probability distribution function, and the latter the cumulative distribution function.

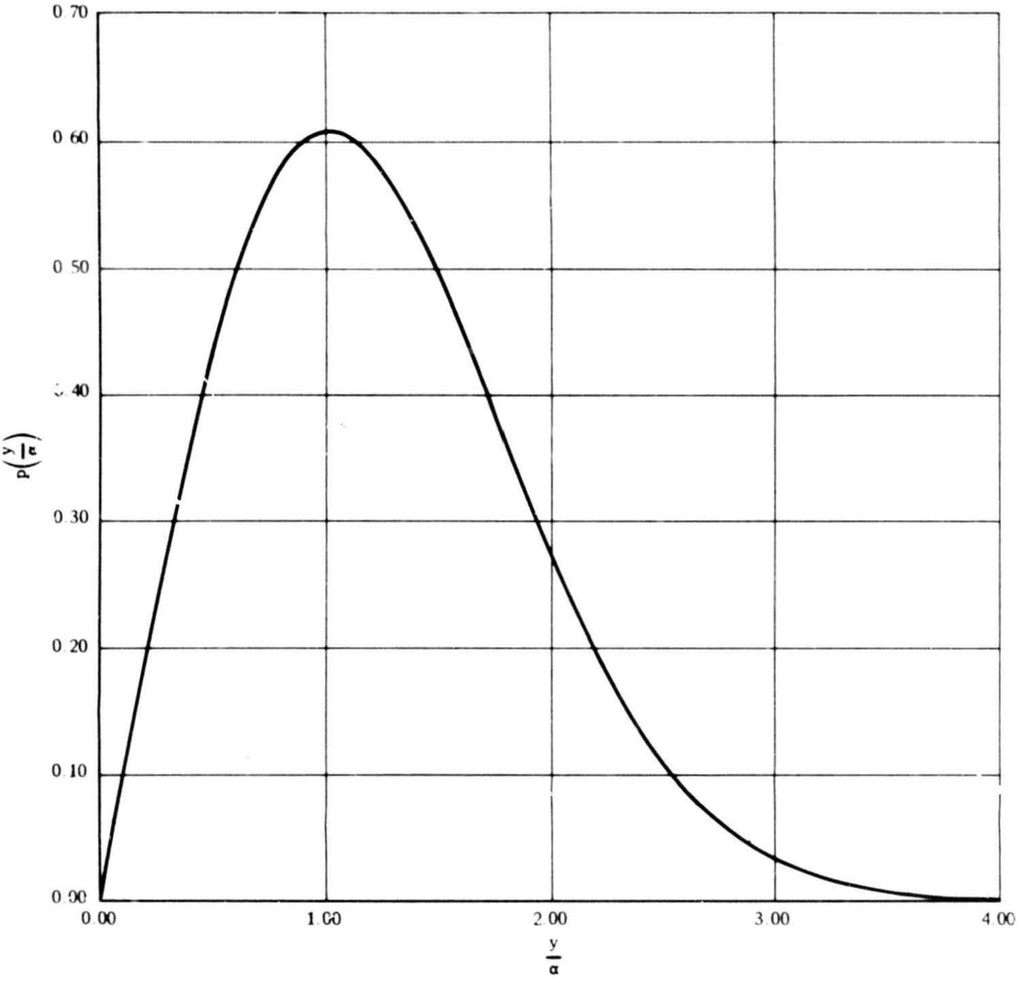

Figure 1. The Rayleigh Density Function.

Table 1. The Rayleigh Density Function $p_R(\mu a)$

μ	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00000	0.01000	0.02000	0.02999	0.03997	0.04994	0.05989	0.06983	0.07974	0.08964
0.1	0.09950	0.10934	0.11914	0.12891	0.13863	0.14832	0.15797	0.16756	0.17711	0.18660
0.2	0.19604	0.20542	0.21474	0.22400	0.23319	0.24231	0.25136	0.26034	0.26924	0.27806
0.3	0.28680	0.29546	0.30403	0.31251	0.32091	0.32921	0.33741	0.34552	0.35353	0.36144
0.4	0.36925	0.37695	0.38454	0.39203	0.39940	0.40667	0.41382	0.42085	0.42777	0.43457
0.5	0.44125	0.44781	0.45424	0.46055	0.46674	0.47280	0.47873	0.48453	0.49021	0.49575
0.6	0.50116	0.50644	0.51159	0.51660	0.52148	0.52622	0.53083	0.53530	0.53963	0.54383
0.7	0.54789	0.55182	0.55560	0.55925	0.56276	0.56613	0.56936	0.57246	0.57542	0.57824
0.8	0.58092	0.58346	0.58587	0.58815	0.59028	0.59228	0.59415	0.59588	0.59748	0.59895
0.9	0.60028	0.60148	0.60255	0.60349	0.60431	0.60499	0.60555	0.60598	0.60629	0.60647
1.0	0.60653	0.60647	0.60629	0.60599	0.60557	0.60504	0.60439	0.60363	0.60276	0.60177
1.1	0.60068	0.59948	0.59818	0.59676	0.59525	0.59364	0.59192	0.59011	0.58820	0.58620
1.2	0.58410	0.58192	0.57964	0.57728	0.57483	0.57229	0.56968	0.56698	0.56420	0.56135
1.3	0.55842	0.55542	0.55235	0.54921	0.54600	0.54273	0.53939	0.53599	0.53253	0.52901
1.4	0.52544	0.52181	0.51812	0.51439	0.51061	0.50678	0.50290	0.49898	0.49502	0.49102
1.5	0.48698	0.48290	0.47879	0.47465	0.47047	0.46627	0.46204	0.45778	0.45349	0.44919
1.6	0.44486	0.44051	0.43615	0.43177	0.42737	0.42296	0.41854	0.41411	0.40967	0.40522
1.7	0.40077	0.39631	0.39185	0.38739	0.38293	0.37846	0.37400	0.36955	0.36510	0.36065
1.8	0.35622	0.35179	0.34737	0.34296	0.33857	0.33418	0.32981	0.32546	0.32112	0.31680
1.9	0.31250	0.30822	0.30396	0.29971	0.29549	0.29129	0.28712	0.28297	0.27884	0.27474
2.0	0.27067	0.26662	0.26261	0.25862	0.25465	0.25072	0.24682	0.24295	0.23911	0.23530
2.1	0.23153	0.22778	0.22407	0.22040	0.21675	0.21315	0.20957	0.20603	0.20253	0.19906
2.2	0.19563	0.19223	0.18887	0.18555	0.18226	0.17901	0.17580	0.17262	0.16948	0.16638
2.3	0.16331	0.16028	0.15729	0.15434	0.15143	0.14855	0.14571	0.14291	0.14014	0.13741
2.4	0.13472	0.13207	0.12945	0.12687	0.12433	0.12183	0.11936	0.11692	0.11453	0.11217
2.5	0.10984	0.10755	0.10530	0.10308	0.10090	0.09875	0.09664	0.09456	0.09251	0.09050
2.6	0.08852	0.08658	0.08467	0.08279	0.08094	0.07913	0.07735	0.07559	0.07387	0.07219
2.7	0.07053	0.06890	0.06730	0.06573	0.06419	0.06268	0.06120	0.05975	0.05832	0.05693
2.8	0.05556	0.05421	0.05289	0.05160	0.05034	0.04910	0.04788	0.04669	0.04553	0.04439
2.9	0.04327	0.04218	0.04111	0.04006	0.03903	0.03803	0.03705	0.03608	0.03514	0.03423
3.0	0.03333	0.03245	0.03159	0.03075	0.02993	0.02913	0.02834	0.02758	0.02683	0.02610
3.1	0.02538	0.02469	0.02401	0.02334	0.02270	0.02206	0.02145	0.02084	0.02026	0.01968
3.2	0.01912	0.01858	0.01805	0.01753	0.01702	0.01653	0.01605	0.01558	0.01513	0.01468
3.3	0.01425	0.01383	0.01342	0.01302	0.01263	0.01255	0.01188	0.01152	0.01117	0.01083
3.4	0.01050	0.01018	0.00987	0.00956	0.00927	0.00898	0.00870	0.00843	0.00816	0.00791
3.5	0.00766	0.00741	0.00718	0.00695	0.00673	0.00651	0.00630	0.00610	0.00590	0.00571
3.6	0.00552	0.00534	0.00517	0.00500	0.00483	0.00467	0.00452	0.00436	0.00422	0.00408
3.7	0.00394	0.00381	0.00368	0.00355	0.00343	0.00331	0.00320	0.00309	0.00298	0.00288
3.8	0.00278	0.00268	0.00259	0.00250	0.00241	0.00233	0.00224	0.00217	0.00209	0.00201
3.9	0.00194	0.00187	0.00181	0.00174	0.00168	0.00162	0.00156	0.00150	0.00145	0.00139
4.0	0.00134	0.00129	0.00124	0.00120	0.00115	0.00111	0.00107	0.00103	0.00099	0.00095

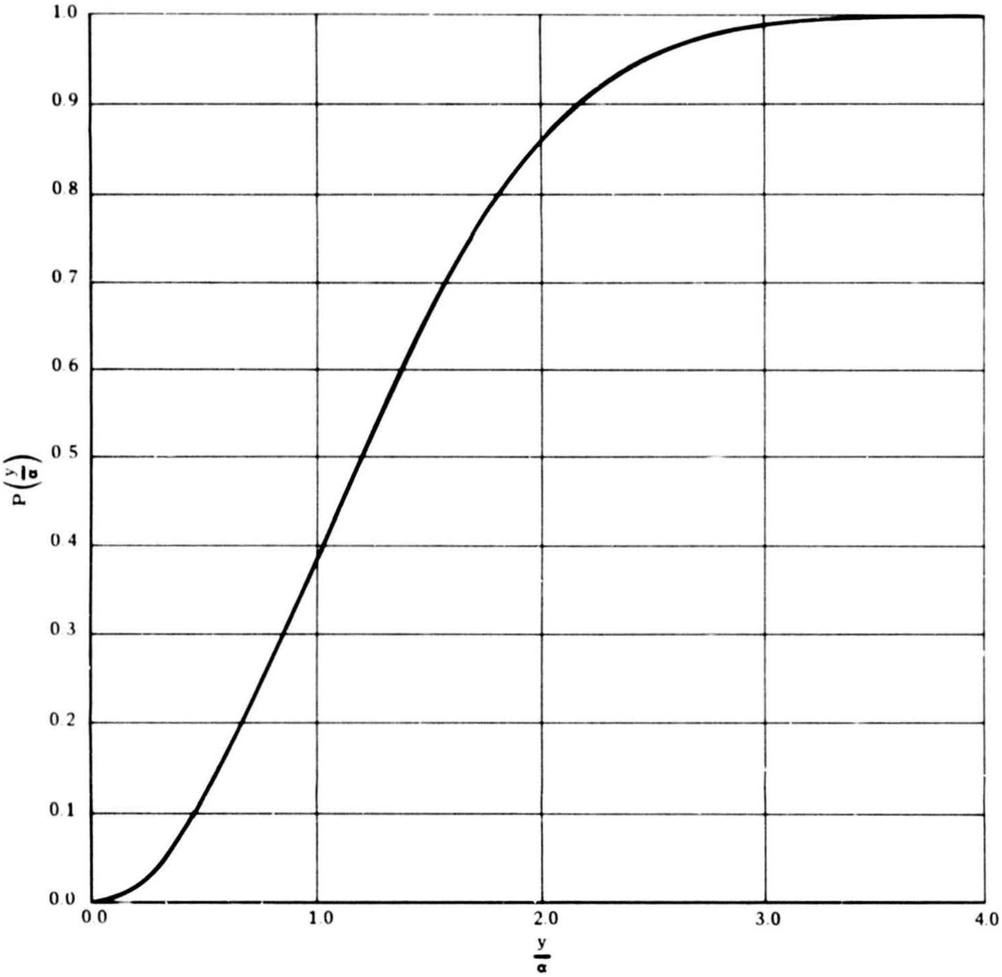

Figure 2. The Rayleigh Distribution Function.

Table 2. The Rayleigh Distribution Function $P_R(\mu a)$

μ	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00000	0.00005	0.00020	0.00045	0.00080	0.00125	0.00180	0.00245	0.00319	0.00404
0.1	0.00499	0.00603	0.00717	0.00841	0.00975	0.01119	0.01272	0.01435	0.01607	0.01789
0.2	0.01980	0.02181	0.02391	0.02610	0.02839	0.03077	0.03324	0.03579	0.03844	0.04118
0.3	0.04400	0.04691	0.04991	0.05299	0.05616	0.05941	0.06275	0.06616	0.06966	0.07323
0.4	0.07688	0.08061	0.08442	0.08831	0.09226	0.09629	0.10040	0.10457	0.10881	0.11312
0.5	0.11750	0.12195	0.12646	0.13103	0.13567	0.14037	0.14512	0.14994	0.15482	0.15975
0.6	0.16473	0.16977	0.17486	0.18000	0.18519	0.19043	0.19571	0.20104	0.20642	0.21184
0.7	0.21730	0.22279	0.22833	0.23391	0.23952	0.24516	0.25084	0.25655	0.26229	0.26806
0.8	0.27385	0.27967	0.28552	0.29139	0.29728	0.30320	0.30913	0.31508	0.32104	0.32703
0.9	0.33302	0.33903	0.34505	0.35108	0.35712	0.36317	0.36922	0.37528	0.38134	0.38740
1.0	0.39347	0.39953	0.40560	0.41166	0.41772	0.42377	0.42982	0.43586	0.44189	0.44791
1.1	0.45393	0.45993	0.46591	0.47189	0.47785	0.48379	0.48972	0.49563	0.50152	0.50740
1.2	0.51325	0.51908	0.52489	0.53067	0.53643	0.54217	0.54788	0.55356	0.55922	0.56484
1.3	0.57044	0.57601	0.58155	0.58706	0.59253	0.59798	0.60339	0.60877	0.61411	0.61942
1.4	0.62469	0.62993	0.63512	0.64029	0.64541	0.65050	0.65555	0.66056	0.66553	0.67046
1.5	0.67535	0.68020	0.68501	0.68977	0.69450	0.69918	0.70382	0.70842	0.71298	0.71749
1.6	0.72196	0.72639	0.73077	0.73511	0.73941	0.74366	0.74787	0.75203	0.75615	0.76022
1.7	0.76425	0.76824	0.77218	0.77608	0.77993	0.78373	0.78750	0.79121	0.79489	0.79852
1.8	0.80210	0.80564	0.80914	0.81259	0.81600	0.81936	0.82268	0.82596	0.82919	0.83238
1.9	0.83553	0.83863	0.84169	0.84471	0.84768	0.85062	0.85351	0.85636	0.85917	0.86194
2.0	0.86466	0.86735	0.87000	0.87260	0.87517	0.87770	0.88018	0.88263	0.88504	0.88742
2.1	0.88975	0.89205	0.89431	0.89653	0.89871	0.90086	0.90298	0.90505	0.90710	0.90910
2.2	0.91108	0.91302	0.91492	0.91680	0.91863	0.92044	0.92221	0.92396	0.92567	0.92735
2.3	0.92899	0.93061	0.93220	0.93376	0.93529	0.93679	0.93826	0.93970	0.94112	0.94250
2.4	0.94387	0.94520	0.94651	0.94779	0.94904	0.95028	0.95148	0.95266	0.95382	0.95495
2.5	0.95606	0.95715	0.95821	0.95926	0.96028	0.96127	0.96225	0.96321	0.96414	0.96506
2.6	0.96595	0.96683	0.96768	0.96852	0.96934	0.97014	0.97092	0.97169	0.97243	0.97317
2.7	0.97388	0.97458	0.97526	0.97592	0.97657	0.97721	0.97783	0.97843	0.97902	0.97960
2.8	0.98016	0.98071	0.98124	0.98177	0.98228	0.98277	0.98326	0.98375	0.98419	0.98464
2.9	0.98508	0.98551	0.98592	0.98633	0.98672	0.98711	0.98748	0.98785	0.98821	0.98855
3.0	0.98889	0.98922	0.98954	0.98985	0.99016	0.99045	0.99074	0.99102	0.99129	0.99155
3.1	0.99181	0.99206	0.99231	0.99254	0.99277	0.99300	0.99321	0.99342	0.99363	0.99383
3.2	0.99402	0.99421	0.99440	0.99457	0.99475	0.99491	0.99508	0.99523	0.99539	0.99554
3.3	0.99568	0.99582	0.99596	0.99609	0.99622	0.99634	0.99646	0.99658	0.99669	0.99680
3.4	0.99691	0.99701	0.99711	0.99721	0.99731	0.99740	0.99749	0.99757	0.99765	0.99773
3.5	0.99781	0.99789	0.99796	0.99803	0.99810	0.99817	0.99823	0.99829	0.99835	0.99841
3.6	0.99847	0.99852	0.99857	0.99862	0.99867	0.99872	0.99877	0.99881	0.99885	0.99890
3.7	0.99894	0.99897	0.99901	0.99905	0.99908	0.99912	0.99915	0.99918	0.99921	0.99924
3.8	0.99927	0.99930	0.99932	0.99935	0.99937	0.99940	0.99942	0.99944	0.99946	0.99948
3.9	0.99950	0.99952	0.99954	0.99956	0.99957	0.99959	0.99961	0.99962	0.99964	0.99965
4.0	0.99966	0.99968	0.99969	0.99970	0.99971	0.99973	0.99974	0.99975	0.99976	0.99977

Maximum Value of Rayleigh Density Function

The maximum value of the Rayleigh density function which will occur when $x = \alpha$ in equation (1) is

$$\max_{0 \leq x < \infty} \{p_R(x)\} = p_R(\alpha) = \frac{1}{\alpha} e^{-\frac{1}{2}} \approx \frac{1}{\alpha}(0.60653) \tag{3}$$

Mean of Rayleigh Probability Density Function

The mean of a random variable which has a Rayleigh density function is given by

$$\mu_R = \sqrt{\frac{\pi}{2}}\,\alpha = \alpha(1.2533) \tag{4}$$

This relationship is illustrated in figure 3.

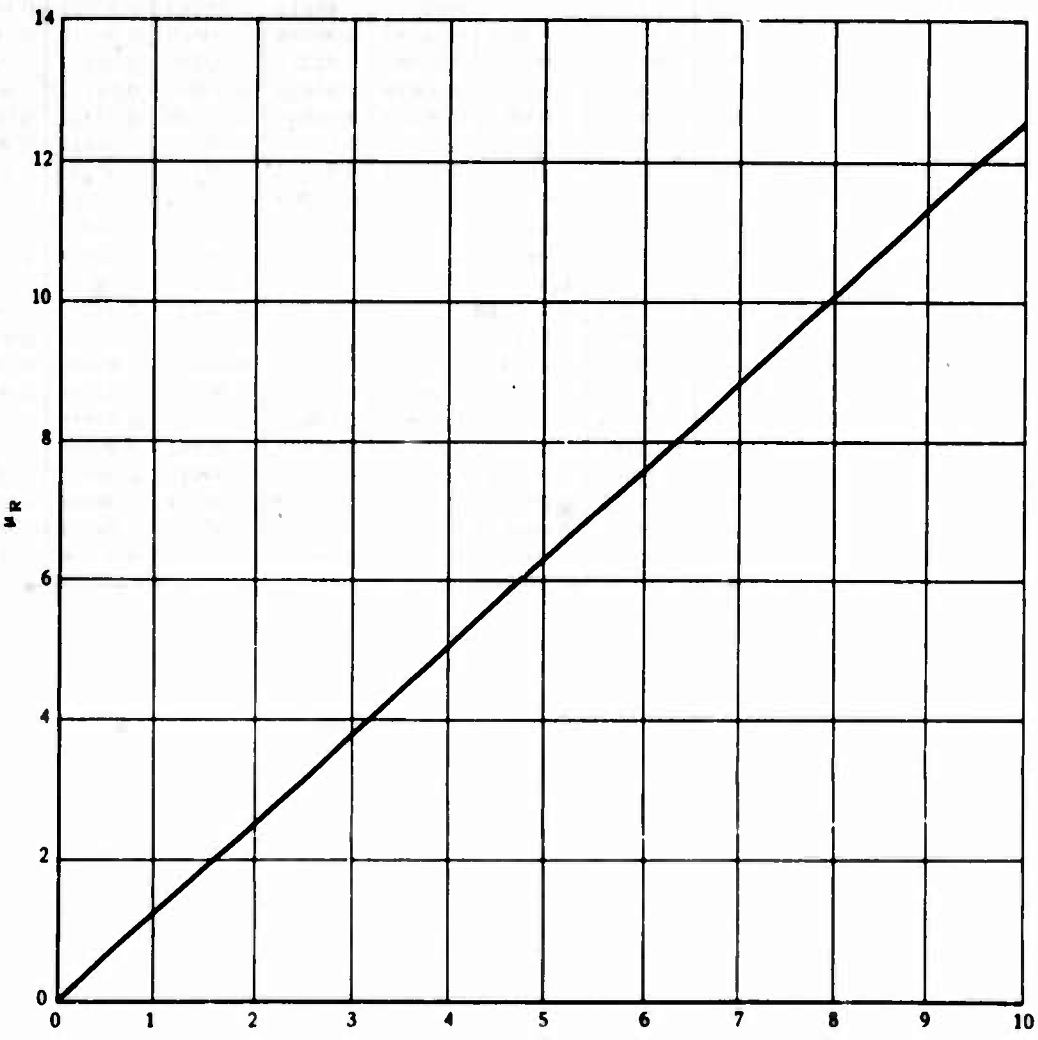

Figure 3. The Relationship Between α, the Standard Parameter of the Rayleigh Density Function, and the Mean of the Rayleigh Density Function.

Mean Square of Rayleigh Probability Density Function

The mean square of a random variable X which has a Rayleigh density function is given by

$$E(X^2) = 2a^2 \tag{5}$$

Standard Deviation of Rayleigh Probability Density Function

The standard deviation of a random variable which has a Rayleigh density function is given by

$$\sigma_R = a\sqrt{2 - \frac{\pi}{2}} \approx 0.65514a \tag{6}$$

This relationship is illustrated in figure 4.

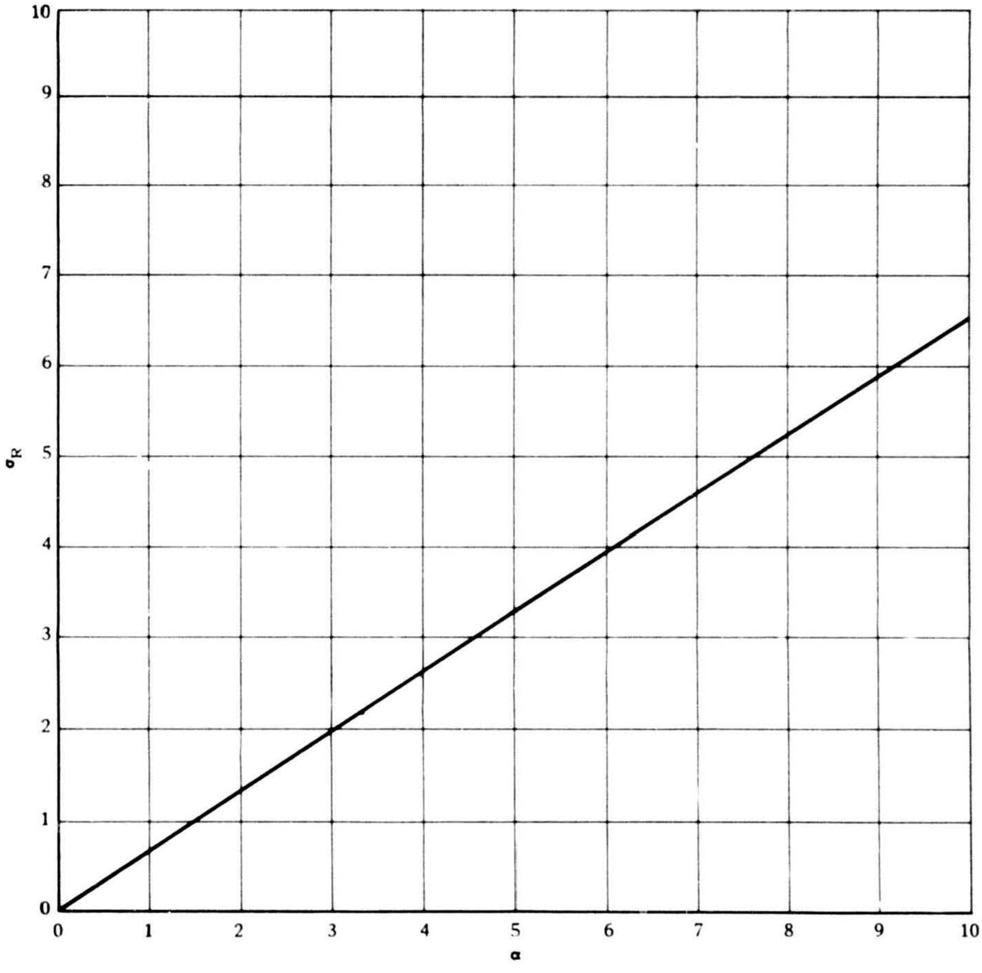

Figure 4. The Relationship Between a, the Standard Parameter of the Rayleigh Density Function, and the Standard Deviation of the Rayleigh Density Function.

RELATIONSHIP BETWEEN THE RAYLEIGH DENSITY AND SOME WELL KNOWN PROBABILITY DENSITIES

If X and Y are independent random variables which have normal density functions with respective means μ_X, μ_Y and respective standard deviations σ_X, σ_Y such that $\sigma_X = \sigma_Y$ and if

$$R = \sqrt{\left(\frac{X - \mu_X}{\sigma_X}\right)^2 + \left(\frac{Y - \mu_Y}{\sigma_Y}\right)^2}$$

then R is Rayleigh distributed with parameter $\alpha = 1$. The random variable R is also said to have a circular normal density.

If a random variable X has a chi squared density function with two degrees of freedom, then $\sqrt{X}$ has a Rayleigh density function.

JOINT PROBABILITY DENSITY FUNCTION OF TWO INDEPENDENT RAYLEIGH DISTRIBUTED RANDOM VARIABLES

The joint probability density function of two independent Rayleigh distributed random variables X_1 and X_2 with respective parameters α_1 and α_2 is given by

$$p_{X_1,X_2}(x_1,x_2) = \begin{cases} 0 & \text{if } x_1 < 0 \text{ or } x_2 < 0 \\ \dfrac{x_1 x_2}{\alpha_1^2 \alpha_2^2} \exp\left[-\left(\dfrac{x_1^2}{2\alpha_1^2} + \dfrac{x_2^2}{2\alpha_2^2}\right)\right] & \text{if } x_1 \geq 0 \text{ and } x_2 \geq 0 \end{cases} \qquad (7)$$

This function is illustrated in figure 5.

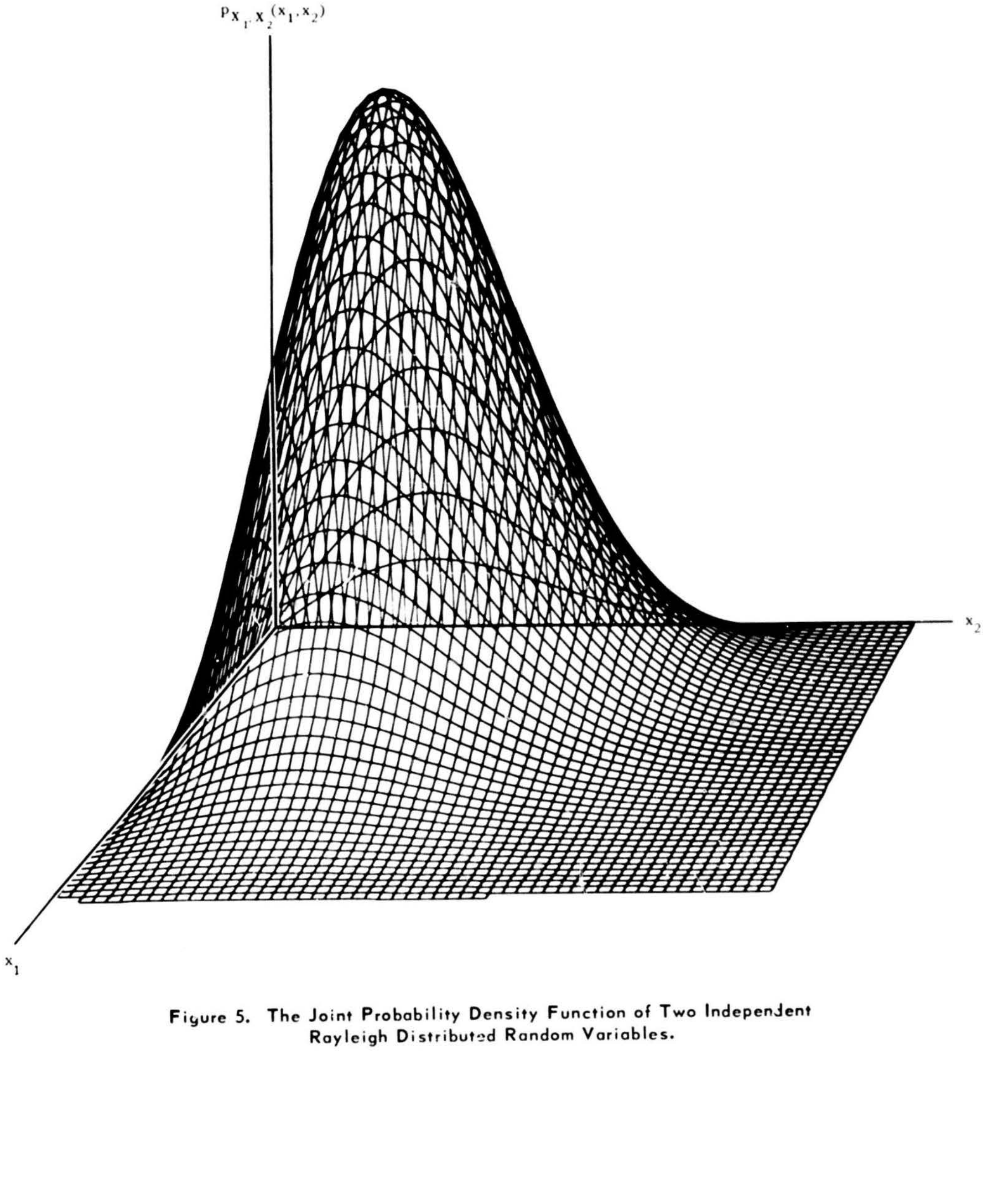

Figure 5. The Joint Probability Density Function of Two Independent Rayleigh Distributed Random Variables.

SUM OF TWO RAYLEIGH DISTRIBUTED RANDOM VARIABLES

Density Function for Sum of Two Independent Rayleigh Distributed Random Variables

If X_1 and X_2 are two independent random variables with Rayleigh density functions and respective parameters α_1 and α_2, then the density function of the random variable $Y = X_1 + X_2$ is given by

$$p_Y(y) = 0 \quad \text{if } y < 0$$

and if $y \geq 0$ it is given by

$$p_Y(y) = \frac{\alpha_1^2 y}{(\alpha_1^2 + \alpha_2^2)^2} \exp\left(-\frac{y^2}{2\alpha_1^2}\right) + \frac{\alpha_2^2 y}{(\alpha_1^2 + \alpha_2^2)} \exp\left(-\frac{y^2}{2\alpha_2^2}\right)$$

$$+ \sqrt{\frac{\pi}{2}} \frac{\alpha_1\alpha_2[y^2 - (\alpha_1^2 + \alpha_2^2)]}{(\alpha_1^2 + \alpha_2^2)^{\frac{5}{2}}} \exp\left[-\frac{y^2}{2(\alpha_1^2 + \alpha_2^2)}\right] \tag{8}$$

$$\left\{\operatorname{erf}\left[\frac{y\alpha_2}{\alpha_1\sqrt{2(\alpha_1^2 + \alpha_2^2)}}\right] + \operatorname{erf}\left[\frac{y\alpha_1}{\alpha_2\sqrt{2(\alpha_1^2 + \alpha_2^2)}}\right]\right\}$$

where

$$\operatorname{erf}(\theta) = \frac{2}{\sqrt{\pi}} \int_0^\theta e^{-t^2}\, dt$$

is the error function. The error function is extensively tabulated and very good numerical approximations exist.* Equation (8) is shown in figure 6 in terms of the standard deviation of Y.** The distribution function for the above density function is shown in figure 7.

Mean and Standard Deviation of Sum of Two Rayleigh Distributed Random Variables

If Y is the random variable as in the density function of equation (8) above, then the mean of Y is given by

$$\mu_Y = \sqrt{\frac{\pi}{2}}(\alpha_1 + \alpha_2) \tag{9}$$

The standard deviation of Y is given by

$$\sigma_Y = \sqrt{\left(2 - \frac{\pi}{2}\right)(\alpha_1^2 + \alpha_2^2)} = 0.65514\sqrt{\alpha_1^2 + \alpha_2^2} \tag{10}$$

* See National Bureau of Standards Applied Mathematics Series 55, Handbook of Mathematical Functions, chapter 7.

** Note that in figure 6, a variable which is the sum of two Rayleigh variables could easily be mistaken for a normally distributed random variable. The only difference is the apparent skewness of the curve. Also note that a variable which is the product of two Rayleigh variables could easily be mistaken for a single Rayleigh distributed random ·iable (see figure 1).

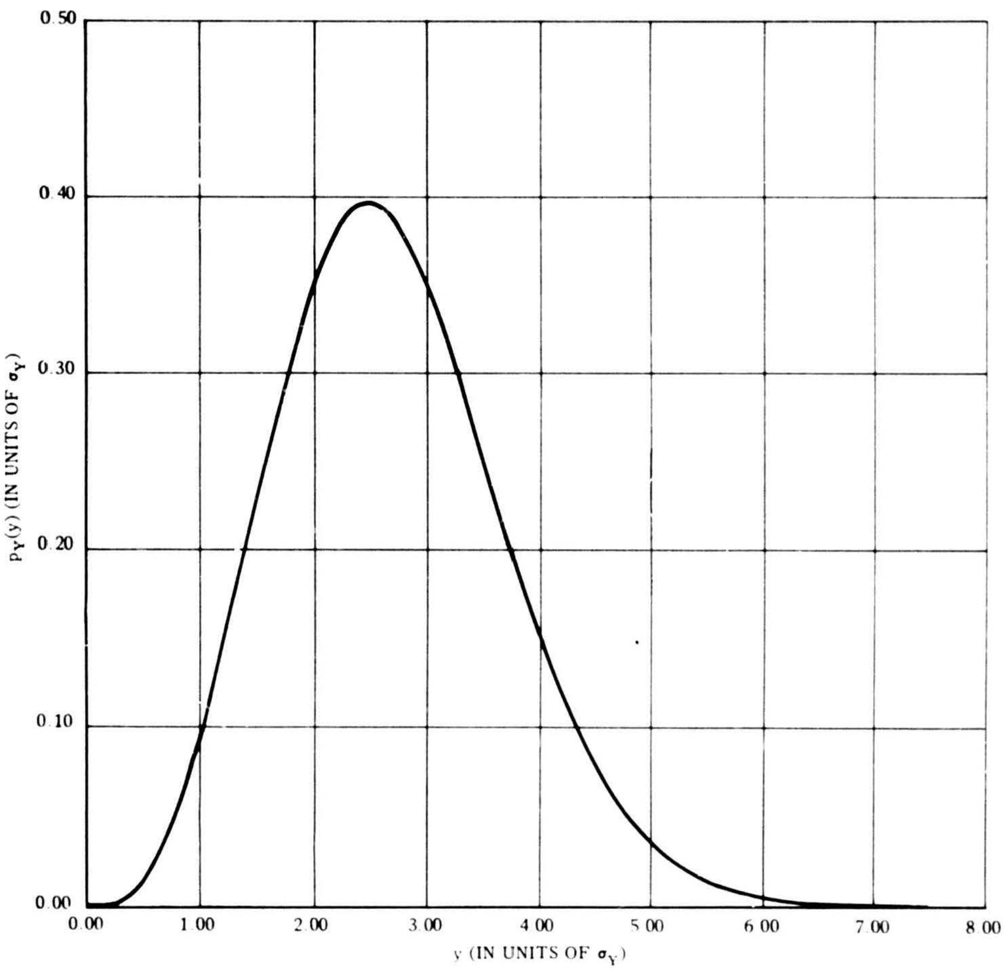

Figure 6. The Density Function of the Sum of Two Independent Rayleigh Distributed Random Variables With Equal Parameters.

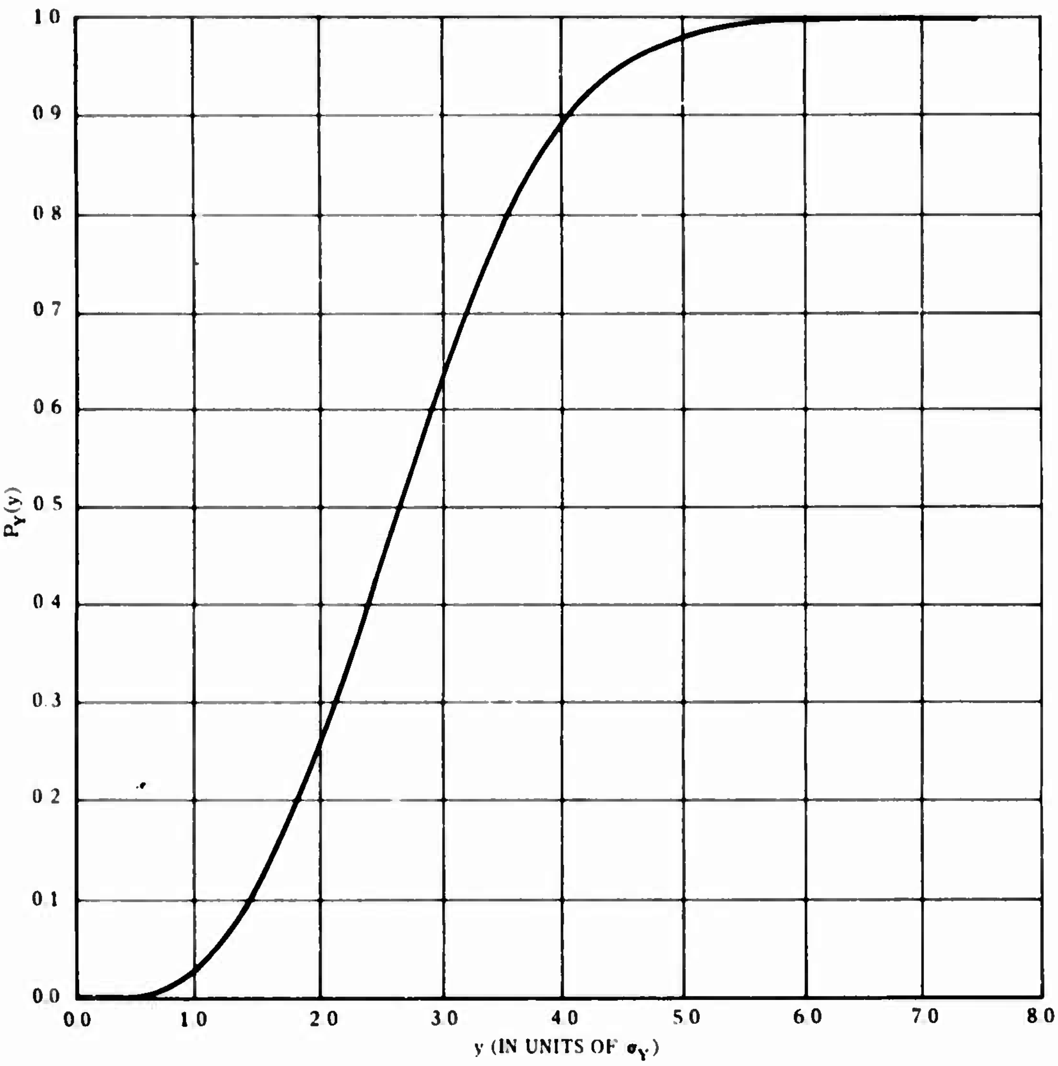

Figure 7. The Distribution Function of the Sum of Two Independent Rayleigh Distributed Random Variables With Equal Parameters.

PRODUCT OF TWO RAYLEIGH DISTRIBUTED RANDOM VARIABLES

Density Function for Product of Two Independent Rayleigh Distributed Random Variables

If $Y = X_1X_2$ where X_1 and X_2 are independent random variables with Rayleigh density functions with respective parameters α_1 and α_2, then the density function of Y is given by

$$p_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \lim\limits_{h \downarrow 0} \displaystyle\int_h^\infty \frac{y}{x\alpha_1^2\alpha_2^2} \exp\left[-\frac{(x^4\alpha_2^2 + \alpha_1^2 y^2)}{2x^2\alpha_1^2\alpha_2^2}\right] dx & \text{if } y \geq 0 \end{cases} \tag{11}$$

This function is numerically integrated and then illustrated in figure 8. The distribution function for the above density function is illustrated in figure 9.

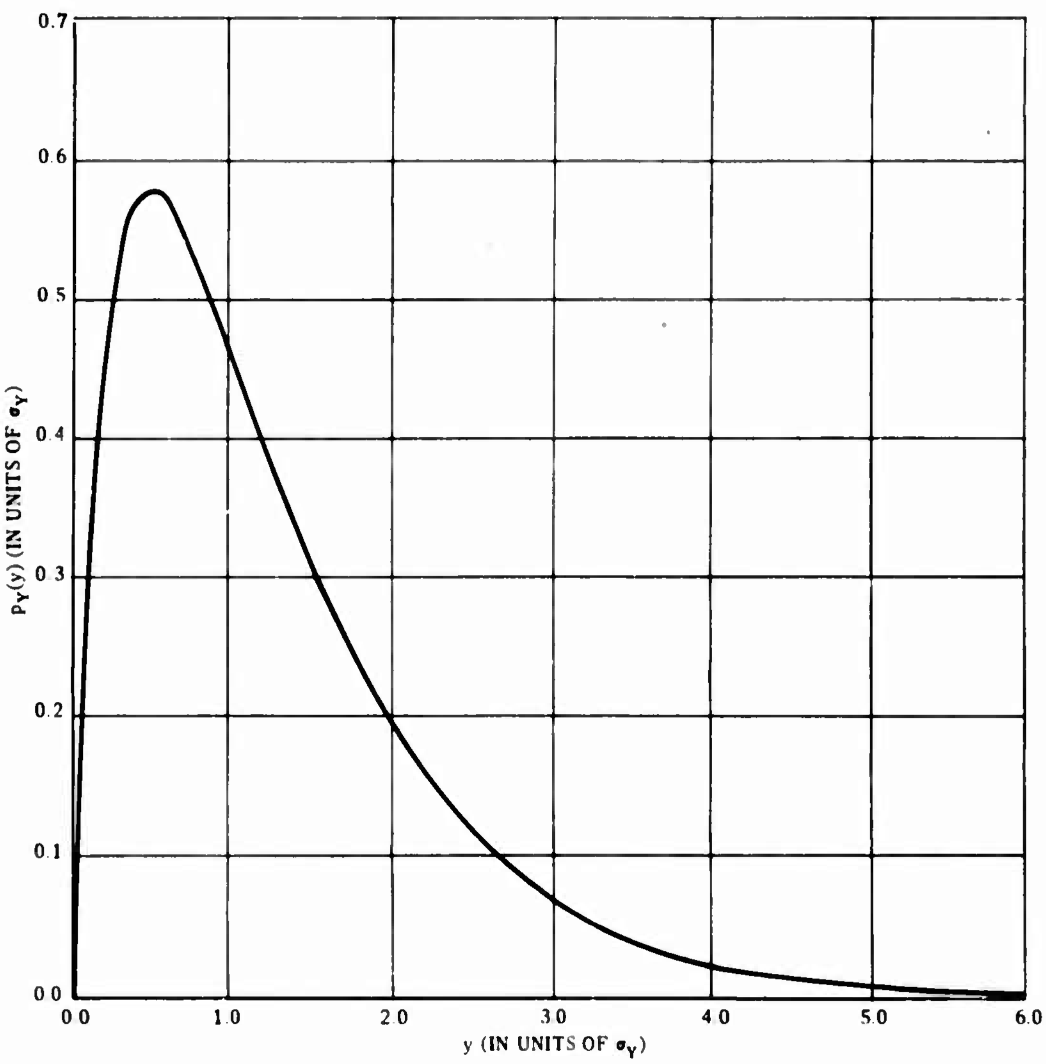

Figure 8. The Density Function for the Product of Two Independent Rayleigh Distributed Random Variables With Equal Parameters.

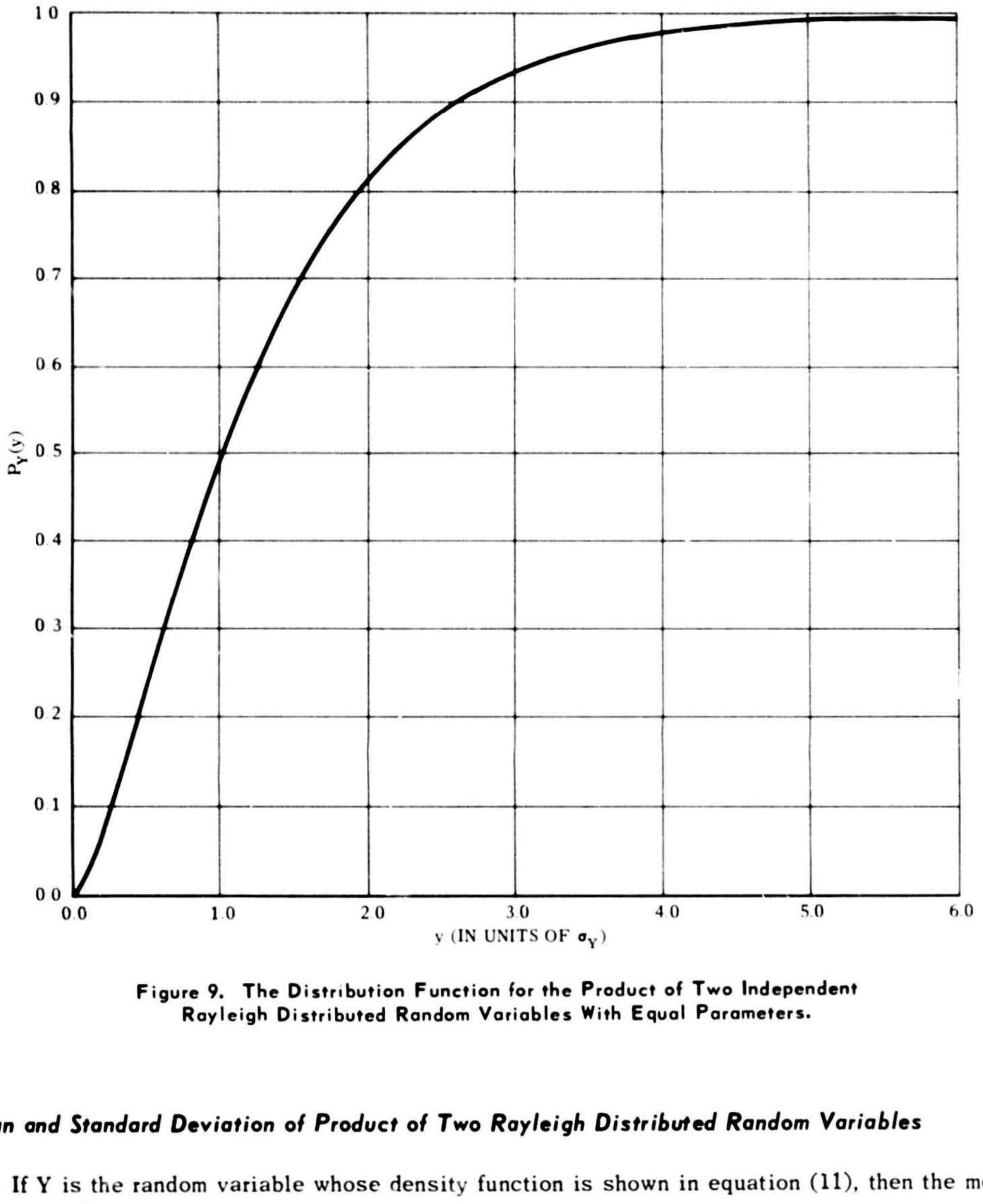

Figure 9. The Distribution Function for the Product of Two Independent Rayleigh Distributed Random Variables With Equal Parameters.

Mean and Standard Deviation of Product of Two Rayleigh Distributed Random Variables

If Y is the random variable whose density function is shown in equation (11), then the mean of Y is given by

$$\mu_Y = \frac{\pi}{2}\alpha_1\alpha_2 \approx 1.5708\alpha_1\alpha_2 \tag{12}$$

The standard deviation of Y is given by

$$\sigma_Y = \alpha_1\alpha_2\sqrt{4 - \frac{\pi^2}{4}} \approx 1.2380\alpha_1\alpha_2 \tag{13}$$

EXAMPLES

Example 1. The use of table 1:

Suppose it is wished to know the ordinate of the Rayleigh density function at 2.14α. Then, using table 1 and the notation of equation (1),

$$p_R(2.14\alpha) = \frac{0.21675}{\alpha}$$

Example 2. The use of table 2:

Suppose one knows that he has a Rayleigh distributed random variable X and wishes to find the probability that $X \leq 1.67\alpha$. Then, using table 2 and the notation of equation (2),

$$P_R(1.67\alpha) = 0.75203$$

Example 3. The use of figures 1, 2, and 3:

Suppose a record of a probability density function which strongly resembles figure 1 requires identification. A Rayleigh density function is then suspect and could be verified if a good estimate of the Rayleigh parameter α was known. Since the value of α is determined by the abscissa value at which the maximum value of the density function occurs, one can determine α immediately. The value of α is thus known and the probability density can be obtained from equation (1) and compared to the actual record. If a quick estimate of the mean and standard deviation is desired, they can be obtained by equations (4) and (6) or by figures 3 and 4 as follows. Suppose $\alpha = 8.5$, then by figure 3 $\mu_R = 10.7$ and by figure 4 $\sigma_R = 5.6$.

Example 4:

The set of scalar miss distances of the AGM Splash 83K in table 3 is considered. It is desired to characterize the population from which this set came. If it is assumed that the x and y coordinates of this set of miss distances are independent, then this set is Rayleigh distributed, and the population from which it came can be completely characterized. The sample mean is computed in the usual manner.

$$\bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$$

where

n The sample size. In this case $n = 30$.

r_i A data point, i.e., one of the given set of scalar miss distances.

$\bar{r}$ The sample mean.

It is found that

$$\bar{r} = 24.9$$

Now from equation (4) or figure 3, an approximation for the α for this population can be found.

$$\alpha \approx \frac{\bar{r}}{1.2533} = 19.9$$

Table 3. Data for Example 4

Scalar Miss Distances for the AGM Splash 83K	
Missile	**Miss Distance (Feet)**
1	5.57
2	7.61
3	8.43
4	9.84
5	10.40
6	10.70
7	13.00
8	13.70
9	15.60
10	16.50
11	18.50
12	19.00
13	19.00
14	19.50
15	19.70
16	20.10
17	22.50
18	22.70
19	27.60
20	29.80
21	30.00
22	31.00
23	32.10
24	32.20
25	38.00
26	40.30
27	40.30
28	45.30
29	60.20
30	68.70

Hence, the probability density for the population can be estimated as

$$p_R(r) = \frac{r}{(19.9)^2} \exp\left[-\frac{r^2}{2(19.9)^2}\right]$$

It is now a simple matter to obtain some percentiles for the population. From table 2 find for example:

$$\text{Probability}[R \leq 1.18\sigma] = \text{Probability}[R \leq 23.5] = 0.50$$

$$\text{Probability}[R \leq 1.67\sigma] = \text{Probability}[R \leq 33.2] = 0.75$$

$$\text{Probability}[R \leq 2.15\sigma] = \text{Probability}[R \leq 42.8] = 0.90$$

Probability[R $\leq$ 2.45a] = Probability[R $\leq$ 48.8] = 0.95

Probability[R $\leq$ 3.04a] = Probability[R $\leq$ 60.5] = 0.99

Probability[R $\geq$ 1.49a = 29.7] = 1.0 - 0.67 = 0.33

Probability[5.0 = .25a $\leq$ R $\leq$ 1.75a = 34.8] = 0.78 - 0.03 = 0.75

Example 5:

Table 4 contains speeds in feet per second of the FIZZ 42A rocket at 3 seconds after launch. This data is assumed to be Rayleigh distributed. The sample average is computed as

$$\overline{x} = \frac{1}{20}\sum_{i=1}^{20} s_i$$

where s_i is a speed for each i = 1, 2, ..., 20

Hence,

$\overline{x}$ = 3,295

Table 4. Data for Example 5

Speed of the FIZZ 42A Rocket 3 Seconds After Launch	
Rocket	Speed (Feet Per Second)
1	559
2	673
3	703
4	1,737
5	1,819
6	1,895
7	2,029
8	2,309
9	2,782
10	3,191
11	3,342
12	3,712
13	4,021
14	4,065
15	4,159
16	4,397
17	5,455
18	5,764
19	5,889
20	7,393

Then α can be obtained by referring to either figure 3 or equation (4). It is found to be

$$\alpha = 2{,}629$$

Therefore, the probability density for the population from which our sample came can be estimated as

$$p_R(s) = \frac{s}{(2{,}629)^2} \exp\left[-\frac{s^2}{2(2{,}629)^2}\right]$$

Also the percentiles can be easily obtained from table 2. For example:

$$\text{Probability}[S \leq 1.18\alpha = 3{,}102] = 0.50$$

$$\text{Probability}[S \leq 2.15\alpha = 5{,}652] = 0.90$$

Example 6:

Table 5 contains data of random sums of two sets of 25 miss distances for the AGM Splash 86. The theory developed for the random sum of two Rayleigh distributed variables can be utilized in this instance.

The sample mean is computed in the usual manner as

$$\bar{r}_s = \frac{1}{25}\sum_{i=1}^{25} r_{s_i} = 75.3$$

where each r_{s_i} is a data point from table 5. Equation (9) is now used to approximate the sum, $\alpha_1 + \alpha_2$, of the original parameters of the populations of the sets of miss distances for the AGM Splash 86. It will be assumed that these populations are identical, hence set $\alpha = \alpha_1 = \alpha_2$ then

$$\alpha \approx \frac{\bar{r}_s}{2\sqrt{\frac{\pi}{2}}} \approx \frac{75.3}{2.5066} = 30.0$$

Now, by the use of equation (10), the standard deviation of the sum of two equal Rayleigh populations can be approximated.

$$\sigma_s \approx \sqrt{2}\,\alpha(0.65514) = 28.8$$

Hence, by the use of figure 7, various percentiles for the population, which consists of random sums of two Rayleigh populations, can be approximated. For example:

$$\text{Probability}[R_s \leq 2.65\sigma = 76.3] = 0.50$$

$$\text{Probability}[R_s \leq 4.05\sigma = 117.] = 0.90$$

Example 7:

Table 6 contains data which are random products of two sets of 25 miss distances for the AGM Splash 86. For this case, the theory developed for the random product of two Rayleigh distributed variables can be used.

Table 5. Data for Example 6

Random Sums of Miss Distances for the AGM Splash 86	
Run	Sum (Feet)
1	13.5
2	21.0
3	42.5
4	42.6
5	49.8
6	50.0
7	50.5
8	55.3
9	55.6
10	55.7
11	64.0
12	64.9
13	71.5
14	80.9
15	81.5
16	86.2
17	87.3
18	90.2
19	91.7
20	93.3
21	102.8
22	122.5
23	128.4
24	133.2
25	148.1

Table 6. Data for Example 7

Random Products of Miss Distances for the AGM Splash 86	
Run	Product (Feet^2)
1	9
2	43
3	409
4	435
5	447
6	588
7	630
8	737
9	760
10	762
11	918
12	1,022
13	1,219
14	1,529
15	1,607
16	1,660
17	1,855
18	2,025
19	2,035
20	2,172
21	2,642
22	3,307
23	3,890
24	4,288
25	5,274

The sample mean is computed in the usual manner as

$$\bar{r}_p = \frac{1}{25}\sum_{i=1}^{25} r_{p_i} = 1{,}611$$

where each r_{p_i} is a data point from table 6. As in example 6, it will be assumed that the two original populations were identical with parameter $a = a_1 = a_2$. Using equation (12),

$$a^2 \approx \frac{2}{\pi}\bar{r}_p \approx 1{,}026$$

is obtained. Hence, the estimated standard deviation of the new population (which consists of the random product of two variables, one from each of the two original Rayleigh populations) is

$$\sigma_p \approx 1.2380(1{,}026) \approx 1{,}270$$

Hence, various percentiles may be estimated using figure 9. For example,

$$\text{Probability}[R_p \leq 1.03\sigma_p = 1{,}308] = 0.50$$

$$\text{Probability}[R_p \leq 2.64\sigma_p = 3{,}353] = 0.90$$

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APPENDIX

MATHEMATICAL DERIVATIONS

PROPERTIES FOR ONE RAYLEIGH DISTRIBUTED RANDOM VARIABLE

The Rayleigh density function for a random variable X is defined as

$$p_R(x) = \begin{cases} 0 & \text{if } x < 0 \\ \dfrac{x}{a^2} \exp\left(-\dfrac{x^2}{2a^2}\right) & \text{if } x \geq 0 \end{cases} \tag{1}$$

where a is a convenient parameter. It is emphasized that a is *not* the standard deviation or the mean of the Rayleigh density. It is merely a convenient parameter with which to work. This utility of a will be made evident in the following discussion.

The Rayleigh distribution function is given by

$$P_R(x) = \begin{cases} 0 & \text{if } x < 0 \\ \displaystyle\int_0^x \frac{t}{a^2} \exp\left(-\frac{t^2}{2a^2}\right) dt & \text{if } x \geq 0 \end{cases} \tag{2}$$

Performing the indicated integration

$$P_R(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - \exp\left(-\dfrac{x^2}{2a^2}\right) & \text{if } x \geq 0 \end{cases} \tag{3}$$

is obtained.

A series of calculations for the mean, mean square, and the standard deviation of the Rayleigh density function in terms of a is now given. The mean is given by

$$\mu_R = E(X) = \int_0^\infty x p_R(x)\, dx = \sqrt{\frac{\pi}{2}}\, a \tag{4}$$

The mean square is given by

$$E(X^2) = \int_0^\infty x^2 p_R(x)\, dx = 2a^2$$

The variance is given by

$$\sigma_R^2 = E(X^2) - \mu_R^2 = 2a^2 - \frac{\pi}{2} a^2 = a^2\left(2 - \frac{\pi}{2}\right) \tag{5}$$

Thus we have the following relationship

$$\sigma_R = a \sqrt{2 - \frac{\pi}{2}} \tag{6}$$

for the standard deviation of the Rayleigh distribution in terms of a.

To find the value of x at which the peak or the mode of the Rayleigh density function occurs, the derivative is taken and set equal to zero. Then, the resulting equation is solved.

$$\frac{d}{dx}[p_R(x)] = \frac{x}{a^2} e^{-\frac{x^2}{2a^2}}\left(-\frac{1}{2a^2} \cdot \frac{2x}{1}\right) + \frac{1}{a^2} e^{-\frac{x^2}{2a^2}} = 0$$

Therefore,

$$\frac{x^2}{a^2} = 1$$

And, since $x > 0$,

$$x = a \tag{7}$$

is the value of x at which the maximum occurs.

The maximum value of $p_R(x)$ is given by

$$p_R(a) = \frac{1}{a} \exp\left(-\frac{1}{2}\right) \approx 0.60653\left(\frac{1}{a}\right)$$

from this result one can see immediately the utility of the "convenient" parameter a.

THE SUM OF TWO RAYLEIGH DISTRIBUTED RANDOM VARIABLES

Now that the case of one Rayleigh distributed random variable has been explained, the result of adding together two such independent variables can be considered.

If X_1 and X_2 are independent Rayleigh distributed random variables with respective parameters a_1 and a_2, the stochastic variable $Y = X_1 + X_2$ is considered. It is desired to obtain the density function and associated properties of Y.

The density functions of X_1 and X_2 are denoted by $p_{X_1}(x_1)$ and $p_{X_2}(x_2)$ respectively. Also the distribution functions of X_1 and X_2 are denoted by $P_{X_1}(x_1)$ and $P_{X_2}(x_2)$ respectively. Similarly $p_Y(y)$ and $P_Y(y)$ denote the density and distribution functions of Y. Also $p_{X_1,X_2}(x_1,x_2)$ and $P_{X_1,X_2}(x_1,x_2)$ denote the joint probability density and distribution functions of X_1 and X_2. Then,

$$P_Y(y) = \text{Probability}\{X_1 + X_2 \le y\} = P_{X_1,X_2}[\{(x_1,x_2): x_1 + x_2 \le y\}]$$

$$= \iint\limits_{\{(x_1,x_2): x_1 + x_2 \le y\}} p_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

$$= \int_0^y \left[\int_0^{y-x_1} p_{X_1,X_2}(x_1,x_2)\, dx_2 \right] dx_1$$

The notation Probability$\{X_1 + X_2 \le y\}$ is to be read "the probability that $X_1 + X_2 \le y$." The notation $\{(x_1,x_2): x_1 + x_2 \le y\}$ is to be read "the set of all ordered pairs (x_1,x_2) such that $x_1 + x_2 \le y$."

The transformation

$$x_1 = x_1$$
$$x_2 = t - x_1$$

is now performed. The Jacobian of this transformation is $J(x_1,t) = 1$. Also, $x_2 = 0$ implies $t = x_1$ and $x_2 = y - x_1$, implies $t = y$. Hence,*

$$P_Y(y) = \int_0^y \left[\int_{x_1}^y p_{X_1,X_2}(x_1, t - x_1)\, dt \right] dx_1$$

This last quantity is differentiated with respect to y using the formula of Leibniz.** For simplicity, let

$$F(x,y) = \int_{x_1}^y p_{X_1,X_2}(x_1, t - x_1)\, dt$$

then

$$P_Y(y) = \int_0^y F(x_1,y)\, dx_1$$

Hence, the derivative is

$$\frac{dP_Y(y)}{dy} = F(y,y)\frac{dy}{dy} - F(0,y)\frac{d0}{dy} + \int_0^y \frac{\partial F}{\partial y}(x_1,y)\, dx$$

*Taylor, A. E., Advanced Calculus. Ginn and Company, p. 430.
**Sokolnikoff, I. S. and E. S. Sokolnikoff. Higher Mathematics for Engineers and Physicists. New York, McGraw-Hill, pp. 167-169.

Now note that

$$F(y,y) = \int_y^y p_{X_1,X_2}(y,t-y)\,dt$$

is zero because the integral over one point is zero. Also $\frac{d0}{dy}$ is zero and

$$\frac{\partial F}{\partial y}(x_1,y) = p_{X_1,X_2}(x_1,y-x_1)$$

Hence the density function of Y

$$p_Y(y) = \int_0^y p_{X_1,X_2}(x_1,y-x_1)\,dx_1$$

is obtained. It is now noted that since X_1 and X_2 are independent random variables,

$$p_{X_1,X_2}(x_1,x_2) = p_{X_1}(x_1)p_{X_2}(x_2)$$

Thus,

$$p_Y(y) = \int_0^y p_{X_1}(x_1)p_{X_2}(y-x_1)\,dx_1$$

Substituting the Rayleigh density function for $p_{X_1}(x_1)$ and $p_{X_2}(y-x_1)$ in the above, one obtains

$$p_Y(y) = \frac{1}{\alpha_1^2\alpha_2^2}\int_0^y (xy-x^2)\exp\left\{-\frac{[\alpha_1^2y^2 - 2xy\alpha_1^2 + x^2(\alpha_1^2+\alpha_2^2)]}{2\alpha_1^2\alpha_2^2}\right\}dx$$

The next task is to determine this function in closed form or in terms of known tabulated functions. To facilitate the algebraic manipulation, several substitutions will be utilized. If

$$\gamma = y\alpha_1^2$$

$$\beta = \alpha_1^2y^2$$

$$\omega = \alpha_1^2 + \alpha_2^2$$

$$\psi = \frac{1}{2\alpha_1^2\alpha_2^2}$$

then

$$p_Y(y) = 2\psi e^{-\psi\beta}\int_0^y (xy-x^2)e^{-\psi(\omega x^2 - 2\gamma x)}\,dx$$

If the exponential expression could be changed to the form $e^{-\psi z^2}$, it would be easier to manipulate. Hence, suppose

$$z^2 = (ax + b)^2 = \omega x^2 - 2\gamma x + \delta$$

where δ is some constant. Since $(ax + b)^2 = a^2x^2 + 2abx + b^2$, equating coefficients of x^2, x, and the constant term with a, 2γ, and δ respectively

$$a = \sqrt{\omega}$$

$$b = -\frac{\gamma}{\sqrt{\omega}}$$

are obtained. Therefore, let $z = \sqrt{\omega}\, x - \dfrac{\gamma}{\sqrt{\omega}}$ then

$$p_Y(y) = 2\psi e^{-\psi\beta} e^{\frac{\gamma^2}{\omega}\psi} \int_0^y (xy - x^2) \exp(-\psi z^2)\, dx$$

Also, the constant term can now be consolidated into

$$a_0 = 2\psi \exp\left[-\psi\left(\beta - \frac{\gamma^2}{\omega}\right)\right]$$

The integration can now be accomplished over z, thus

$$dz = \sqrt{\omega}\, dx$$

and $x_1 = 0$ implies $z = -\dfrac{\gamma}{\sqrt{\omega}}$; $x_1 = y$ implies $z = \sqrt{\omega} y - \dfrac{\gamma}{\sqrt{\omega}}$. Also

$$x = \frac{\sqrt{\omega} z + \gamma}{\omega}$$

If the following simple expressions are introduced for the limits of the integral,

$$\zeta_0 = \sqrt{\omega} y - \frac{\gamma}{\sqrt{\omega}}$$

$$k = -\frac{\gamma}{\sqrt{\omega}}$$

then

$$p_Y(y) = a_0 \int_k^{\zeta_0} \left(\frac{\sqrt{\omega} z y + \gamma y}{\omega} - \frac{\omega z^2 + 2\sqrt{\omega} z \gamma + \gamma^2}{\omega^2} \right) e^{-\psi z^2} \frac{dz}{\sqrt{\omega}}$$

$$= a_0 \int_k^{\zeta_0} \left(\frac{zy}{\omega} + \frac{\gamma y}{\omega^{\frac{3}{2}}} - \frac{z^2}{\omega^{\frac{3}{2}}} - \frac{2z\gamma}{\omega^2} - \frac{\gamma^2}{\omega^{\frac{5}{2}}} \right) e^{-\psi z^2} \, dz$$

$$= \frac{a_0}{\omega} \int_k^{\zeta_0} \left[z\left(y - \frac{2\gamma}{\omega}\right) - \frac{z^2}{\omega^{\frac{1}{2}}} + \left(\frac{\gamma y}{\omega^{\frac{1}{2}}} - \frac{\gamma^2}{\omega^{\frac{3}{2}}} \right) \right] e^{-\psi z^2} \, dz$$

Now to consolidate the constant terms, let

$$a_1 = \frac{a_0}{\omega}\left(y - \frac{2\gamma}{\omega}\right)$$

$$a_2 = -\frac{a_0}{\omega^{\frac{3}{2}}}$$

$$a_3 = \frac{a_0}{\omega^{\frac{3}{2}}}\left(\gamma y - \frac{\gamma^2}{\omega}\right)$$

then the resulting integral is

$$p_Y(y) = a_1 \int_k^{\zeta_0} z e^{-\psi z^2} \, dz + a_2 \int_k^{\zeta_0} z^2 e^{-\psi z^2} \, dz + a_3 \int_k^{\zeta_0} e^{-\psi z^2} \, dz$$

This last expression is now recognizable as one in which some of the integrated terms can be solved in a closed form and some of the terms can be solved in a form for which there exists excellent numerical approximations and tabulations. Each of the preceding three integrals will now be considered separately since each one involves a different type of evaluation. For convenience, set

$$I_1 = a_1 \int_k^{\zeta_0} z e^{-\psi z^2} \, dz$$

$$I_2 = a_2 \int_k^{\zeta_0} z^2 e^{-\psi z^2} \, dz$$

$$I_3 = a_3 \int_k^{\zeta_0} e^{-\psi z^2} \, dz$$

Evaluating I_1,

$$I_1 = -\frac{a_1}{2\psi} e^{-\psi z^2}\Big|_k^{\zeta_0} = \frac{a_1}{2\psi}\left(e^{-\psi k^2} - e^{-\psi\zeta_0^2}\right)$$

is obtained. To evaluate I_3, the error function defined by

$$\mathrm{erf}(\theta) = \frac{2}{\sqrt{\pi}}\int_0^{\theta} e^{-t^2}\, dt \qquad \theta \geq 0$$

and the definite integral

$$\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-t^2}\, dt = 1$$

are utilized. Therefore,

$$\int_{\theta}^{\infty} e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}[1 - \mathrm{erf}(\theta)]$$

and

$$\int_{\theta_1}^{\theta_2} e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}[\mathrm{erf}(\theta_2) - \mathrm{erf}(\theta_1)]$$

It is noted that $\mathrm{erf}(-\theta) = -\mathrm{erf}(\theta)$. Now, to obtain I_3 in terms of the error function, let $\zeta = z\sqrt{\psi}$

then

$$I_3 = \frac{a_3}{\sqrt{\psi}}\int_{k\sqrt{\psi}}^{\zeta_0\sqrt{\psi}} e^{-\zeta^2}\, d\zeta$$

$$= \frac{a_3}{\sqrt{\psi}}\frac{\sqrt{\pi}}{2}[\mathrm{erf}(\zeta_0\sqrt{\psi}) - \mathrm{erf}(k\sqrt{\psi})]$$

And, since

$$k\sqrt{\psi} = -\frac{y\alpha_1}{\sqrt{2}\alpha_2\sqrt{\alpha_1^2 + \alpha_2^2}} \leq 0$$

and

$$\zeta_0 = \frac{y\alpha_2^2}{\sqrt{\alpha_1^2 + \alpha_2^2}} \geq 0$$

it follows that

$$I_3 = \frac{a_3}{2}\sqrt{\frac{\pi}{\psi}}\,[\mathrm{erf}(\zeta_0\sqrt{\psi}) + \mathrm{erf}(-k\sqrt{\psi})]$$

Finally, to evaluate

$$I_2 = a_2\int_k^{\zeta_0} z^2 e^{-\psi z^2}\,dz$$

it is necessary to integrate by parts. Thereafter the procedure is similar to that employed in the evaluation of I_3. Hence, if

$$u = z \qquad du = dz$$

$$dv = ze^{-\psi z^2}\,dz \qquad v = -\frac{1}{2\psi}e^{-\psi z^2}$$

then

$$I_2 = a_2\left[-\frac{z}{2\psi}e^{-\psi z^2}\Big|_k^{\zeta_0} + \frac{1}{2\psi}\int_k^{\zeta_0} e^{-\psi z^2}\,dz\right]$$

$$= -\frac{a_2}{2\psi}\zeta_0 e^{-\psi\zeta_0^2} + \frac{a_2 k}{2\psi}e^{-\psi k^2}$$

$$+ \frac{a_2\sqrt{\pi}}{4\psi^{\frac{3}{2}}}[\mathrm{erf}(\zeta_0\sqrt{\psi}) + \mathrm{erf}(-k\sqrt{\psi})]$$

Then, in final form,

$$p_Y(y) = I_1 + I_2 + I_3$$

$$= \frac{a_1}{2\psi}\left(e^{-\psi k^2} - e^{-\psi\zeta_0^2}\right) + \frac{a_3}{2}\sqrt{\frac{\pi}{\psi}}\,[\mathrm{erf}(\zeta_0\sqrt{\psi}) + \mathrm{erf}(-k\sqrt{\psi})]$$

$$- \frac{a_2\zeta_0}{2\psi}e^{-\psi\zeta_0^2} + \frac{a_2 k}{2\psi}e^{-\psi k^2}$$

$$+ \frac{a_2\sqrt{\pi}}{4\psi^{\frac{3}{2}}}\,\mathrm{erf}(\zeta_0\sqrt{\psi}) + \mathrm{erf}(-k\sqrt{\psi})]$$

$$= \frac{(a_1 + a_2 k)}{2\psi}e^{-\psi k^2} - \frac{(a_1 + a_2\zeta_0)}{2\psi}e^{-\psi\zeta_0^2}$$

$$+ \left(a_3 + \frac{a_2}{2\psi}\right)\left(\frac{1}{2}\sqrt{\frac{\pi}{\psi}}\right)[\mathrm{erf}(\zeta_0\sqrt{\psi}) + \mathrm{erf}(-k\sqrt{\psi})]$$

The reader can now substitute back to obtain a form consisting of the original parameters and variables. After these substitutions,

$$p_Y(y) = \frac{\alpha_1^2 y}{(\alpha_1^2 + \alpha_2^2)^2} \exp\left(-\frac{y^2}{2\alpha_1^2}\right) + \frac{\alpha_2^2 y}{(\alpha_1^2 + \alpha_2^2)^2} \exp\left(-\frac{y^2}{2\alpha_2^2}\right)$$

$$+ \frac{\alpha_1 \alpha_2 [y^2 - (\alpha_1^2 + \alpha_2^2)]}{(\alpha_1^2 + \alpha_2^2)^{\frac{5}{2}}} \exp\left[-\frac{y^2}{2(\alpha_1^2 + \alpha_2^2)}\right]$$

$$\sqrt{\frac{\pi}{2}}\left[\operatorname{erf}\left(\frac{y\alpha_2}{\alpha_1\sqrt{2(\alpha_1^2 + \alpha_2^2)}}\right) + \operatorname{erf}\left(\frac{y\alpha_1}{\alpha_2\sqrt{2(\alpha_1^2 + \alpha_2^2)}}\right)\right]$$

This is a convenient form for the density function of the sum of two randomly distributed Rayleigh variables since there exist extensive tables of the error function. There also exists extremely good numerical approximations.*

It is now desired to obtain the mean and standard deviation of $p_Y(y)$ in terms of α_1 and α_2. The means of X_1, X_2, and Y are denoted by μ_{X_1}, μ_{X_2} and μ_Y. Hence

$$\mu_Y = \int_0^\infty y p_Y(y)\, dy = \int_0^\infty \int_0^\infty (x_1 + x_2) p_{X_1}(x_1) p_{X_2}(x_2)\, dx_1 dx_2$$

$$= \int_0^\infty x_1 p_{X_1}(x_1)\, dx_1 + \int_0^\infty x_2 p_{X_2}(x_2)\, dx_2$$

$$= \mu_{X_1} + \mu_{X_2}$$

$$= \sqrt{\frac{\pi}{2}}(\alpha_1 + \alpha_2)$$

Also, if the respective standard deviations of X_1, X_2 and Y are denoted by σ_{X_1}, σ_{X_2}, and σ_Y, then

$$\sigma_Y^2 = \int_0^\infty (y - \mu_Y)^2 p_Y(y)\, dy$$

$$= \sigma_{X_1}^2 + \sigma_{X_2}^2$$

$$= (\alpha_1^2 + \alpha_2^2)\left(2 - \frac{\pi}{2}\right)$$

This completes the analysis of the sum of two independent Rayleigh distributed random variables.

*National Bureau of Standards Applied Mathematics Series 55, Handbook of Mathematical Functions, chapter 7.

THE PRODUCT OF TWO RAYLEIGH DISTRIBUTED RANDOM VARIABLES

It is also possible to consider the case of a product of two Rayleigh distributed variables and obtain some properties of the resulting random variable.

If X_1 and X_2 are independent Rayleigh distributed random variables with respective parameters α_1, α_2, then $Y = X_1 X_2$ is considered. It is desired to find the probability density function of Y and the mean and standard deviation of Y in terms of the original parameters α_1, α_2.

To find the probability density function of Y it is first noted that since X_1 and X_2 are independent random variables, the joint probability density function X_1 and X_2 is given by

$$p_{X_1,X_2}(x_1,x_2) = p_{X_1}(x_1)p_{X_2}(x_2)$$

where $p_{X_1}(x_1)$ and $p_{X_2}(x_2)$ are Rayleigh probability density functions. Now, if $P_Y(y)$ is the probability distribution function of Y, then

$$P_Y(y) = \text{Probability}[X_1 X_2 \leq y] = \text{Probability}[\{(x_1,x_2): x_1 x_2 \leq y\}]$$

$$= \iint\limits_{\{(x_1,x_2): x_1 x_2 \leq y\}} p_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

$$= \lim_{h \downarrow 0} \int_h^\infty dx_1 \int_0^{\frac{y}{x_1}} p_{X_1,X_2}(x_1,x_2)\, dx_2$$

The transformation

$$x_1 = x$$

$$x_2 = \frac{t}{x}$$

is now performed. The Jacobian of this transformation is $J(x,t) = \frac{1}{x}$. Also, $x_2 = 0$ implies $\frac{t}{x} = 0$, which implies $t = 0$, and $x_2 = \frac{y}{x}$ implies $\frac{t}{x} = \frac{y}{x}$ which implies $y = t$. Hence,*

$$P_Y(y) = \lim_{h \downarrow 0} \int_h^\infty dx \int_0^y p_{X_1,X_2}\left(x, \frac{t}{x}\right) \frac{dt}{x}$$

Now differentiating with respect to y, the density function of the random variable Y is obtained as

$$p_Y(y) = \lim_{h \downarrow 0} \int_h^\infty p_{X_1,X_2}\left(x, \frac{y}{x}\right) \frac{dx}{x}$$

$$= \lim_{h \downarrow 0} \int_h^\infty \frac{y}{x\alpha_1^2\alpha_2^2} \exp\left[-\frac{(x^4\alpha_2^2 + \alpha_1^2 y^2)}{2x^2\alpha_1^2\alpha_2^2}\right] dx$$

This result is integrated numerically.

*See National Bureau of Standards Applied Mathematics Series 55, Handbook of Mathematical Functions, chapter 7.

Now, since the variables X_1 and X_2 are independent, the mean and standard deviation of Y are easily obtained. The mean is given by

$$\mu_Y = \int_0^\infty y p_Y(y)\,dy = \int_0^\infty \int_0^\infty x_1 x_2 p_{X_1}(x_1) p_{X_2}(x_2)\,dx_1\,dx_2$$

$$= \int_0^\infty x_1 p_{X_1}(x_1)\,dx_1 \int_0^\infty x_2 p_{X_2}(x_2)\,dx_2 = \mu_{X_1}\mu_{X_2}$$

$$= \frac{\pi}{2} a_1 a_2$$

The mean square is given by

$$E(Y^2) = \int_0^\infty y^2 p_Y(y)\,dy = \int_0^\infty \int_0^\infty x_1^2 x_2^2 p_{X_1}(x_1) p_{X_2}(x_2)\,dx_1\,dx_2$$

$$= \int_0^\infty x_1^2 p_{X_1}(x_1)\,dx_1 \int_0^\infty x_2^2 p_{X_2}(x_2)\,dx_2$$

$$= E(X_1^2)E(X_2^2)$$

$$= 4a_1^2 a_2^2$$

Thus,

$$\sigma_Y = \sqrt{E(Y^2) - \mu_Y^2}$$

$$= a_1 a_2 \sqrt{4 - \frac{\pi^2}{4}}$$

The pertinent features of the product of two independent Rayleigh distributed random variables are now established.